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Distances and Cuts in Planar Graphs

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We prove the following theorem. Let $G=(V, E)$ be a planar bipartite graph, embedded in the euclidean plane. Let O and I be two of its faces. Then there exist pairwise edge-disjoint cuts $C_1, ..., C_t$ so that for each two vertices v, w with $v, w \in O$ or $v, w \in I$, the distance from v to w in G is equal to the number of cuts C_j separating v and w. This theorem is dual to a theorem of Okamura on plane multicommodity flows, in the same way as a theorem of Karzanov is dual to one of Lomonosov. © 1989 Academic Press, Inc.

1. INTRODUCTION

We prove the following theorem:

THEOREM. *Let $G=(V, E)$ be a planar bipartite graph, embedded in the euclidean plane. Let O and I be two of the faces. Then there exist pairwise edge-disjoint cuts $\delta(X_1), ..., \delta(X_t)$ so that for each two vertices v, w with $v, w \in O$ or $v, w \in I$, the distance of v to w in G is equal to the number of cuts $\delta(X_j)$ separating v and w.*

[Here, for $X \subseteq V$, $\delta(X) := \{e \in E \mid |e \cap X| = 1\}$, while $\delta(X)$ *separates* v and w if $|\{v, w\} \cap X| = 1$.]

Note that for any graph G, whatever collection of pairwise edge-disjoint cuts $\delta(X_j)$ we take, for any two vertices v, w of G, the distance from v to w is always at least as large as the number of these cuts separating v and w. The point in the theorem is that we can get equality under the conditions given.

This theorem is "dual" to a theorem of Okamura [9] on plane multicommodity flows, in the same way as the results of Karzanov [4] are dual to those of Lomonosov [6, 7] on multicommodity flows, as we shall explain in Section 2 below. The theorem extends a result of Hurkens,

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Schrijver, and Tardos [3], dual to a theorem of Okamura and Seymour [10]; this result restricts v, w to belong to only one fixed face.

The theorem cannot be generalized to the obvious extension with more than two faces, as is shown by the complete bipartite graph $K_{2,3}$. This graph also shows that we cannot allow in the theorem above pairs v, w with $v \in O$ and $w \in I$.

2. Relation to Multicommodity Flows

In this section we discuss a relation of the theorem above with multicommodity flow problems. Let $G=(V, E)$ be an undirected graph. Let $\{r_1, s_1\}, ..., \{r_k, s_k\}$ be pairs of vertices ($r_i \neq s_i$ for $i=1, ..., k$). Suppose we wish to decide if

> there exist pairwise edge-disjoint paths $P_1, ..., P_k$ so that P_i connects r_i and s_i ($i=1, ..., k$). (1)

Clearly, the following "cut condition" is a necessary condition:

$$\text{each cut } \delta(X) \text{ separates at most } |\delta(X)| \text{ of the pairs } r_i, s_i. \tag{2}$$

Now Lomonosov [6, 7] (extending earlier work by Menger [8], Hu [1], Rothschild and Whinston [12], Papernov [11], Seymour [15]), Okamura [9] (extending earlier work by Okamura and Seymour [10]), and Seymour [16] showed the following three results, each of which uses the following "parity condition":

$$\text{for each vertex } v, |\delta(\{v\})| + |\{i \mid v \in \{r_i, s_i\}\}| \text{ is even.} \tag{3}$$

Lomonosov's theorem. *If*

> *the graph* $H := (\{r_1, s_1, ..., r_k, s_k\}, \{\{r_1, s_1\}, ..., \{r_k, s_k\}\})$ *has at most four vertices, or is isomorphic to* C_5 *(the circuit with five vertices), or contains two vertices* v, w *so that* $\{v, w\} \cap \{r_i, s_i\} \neq \emptyset$ *for all* $i=1, ..., k$, (4)

then the cut condition (2) *and the parity condition* (3) *together imply* (1).

Okamura's theorem. *If*

> *G is planar, so that there are two of its faces, O and I, with for each* $i=1, ..., k: r_i, s_i \in O$ *or* $r_i, s_i \in I$, (5)

then the cut condition (2) *and the parity condition* (3) *together imply* (1).

Seymour's theorem. *If*

$$\textit{the graph } (V, E \cup \{\{r_1, s_1\}, ..., \{r_k, s_k\}\}) \textit{ is planar,} \tag{6}$$

then the cut condition (2) *and the parity condition* (3) *together imply* (1).

A consequence of these results is that if (4), (5), or (6) holds, and if, moreover, the cut condition (2) holds, then there exist paths $P'_1, P''_1, ..., P'_k, P''_k$ so that both P'_i and P''_i connect r_i and s_i $(i = 1, ..., k)$ and so that each edge of G is in at most two of the paths $P'_1, P''_1, ..., P'_k, P''_k$. (This follows by duplicating each edge of G and each pair $\{r_i, s_i\}$, after which (2) and (3) hold.)

Hence, if (4), (5), or (6) holds, and if $c \in \mathbb{Q}^E_+$ (a "capacity function") and $d \in \mathbb{Q}^k_+$ (a "demand function") so that

$$\text{for each } X \subseteq V, \sum (c_e \mid e \in \delta(X)) \geqslant \sum (d_i \mid i = 1, ..., k; X \text{ separates } r_i \text{ and } s_i), \tag{7}$$

then there exist paths $P^1_1, ..., P^{t_1}_1, P^1_2, ..., P^{t_2}_2, ..., P^1_k, ..., P^{t_k}_k$ (where each P^j_i connects r_i and s_i) and rationals $\lambda^1_1, ..., \lambda^{t_1}_1, \lambda^1_2, ..., \lambda^{t_2}_2, ..., \lambda^1_k, ..., \lambda^{t_k}_k \geqslant 0$ so that

$$\begin{aligned} \sum_{i=1}^{k} \sum_{\substack{j=1 \\ e \in P^j_i}}^{t_i} \lambda^j_i \leqslant c_e \qquad & (e \in E), \\ \sum_{j=1}^{t_i} \lambda^j_i = d_i \qquad & (i = 1, ..., k) \end{aligned} \tag{8}$$

(a "multicommodity flow"). (For (5) this is a result of Papernov [11].) (This result follows from the result in the previous paragraph, by observing that we may take, without loss of generality, c and d to be integral; and hence we can replace each edge e of G by c_e parallel edges, and each pair $\{r_i, s_i\}$ by d_i parallel pairs, after which we apply the previous result.)

In polyhedral terms, this statement is equivalent to: if (4), (5), or (6) holds, then the cone of vectors $(d; c) \in \mathbb{Q}^k \times \mathbb{Q}^E$ defined by the linear inequalities

$$\begin{aligned} &\text{(i)} \quad \sum (c_e \mid e \in \delta(X)) \geqslant \sum (d_i \mid i \in \rho(X)) \qquad && (X \subseteq V), \\ &\text{(ii)} \quad d_i \geqslant 0 && (i = 1, ..., k), \\ &\text{(iii)} \quad c_e \geqslant 0 && (e \in E) \end{aligned} \tag{9}$$

(where $\rho(X) := \{i = 1, ..., k \mid X \text{ separates } r_i \text{ and } s_i\}$), is equal to the cone generated by the following vectors:

$$\begin{array}{lll} \text{(i)} & (\varepsilon_i; \chi^P) & (i = 1, ..., k;\ P\ r_i - s_i\text{-path}), \\ \text{(ii)} & (0; \varepsilon_e) & (e \in E). \end{array} \tag{10}$$

(Here ε_i denotes the ith unit basis vector in $\mathbb{Q}^k$; ε_e denotes the eth unit basis vector in $\mathbb{Q}^E$; χ^P is the *incidence vector* of P in $\mathbb{Q}^E$, i.e., $\chi^P(e) = 1$ if $e \in P$ and $= 0$ otherwise.)

By polarity, this last statement is equivalent to: if (4), (5), or (6) holds, then the cone of vectors $(b; l) \in \mathbb{Q}^k \times \mathbb{Q}^E$ defined by the linear inequalities

$$\begin{array}{lll} \text{(i)} & b_i + \sum_{e \in P} l_e \geqslant 0 & (i = 1, ..., k;\ P\ r_i - s_i\text{-path}), \\ \text{(ii)} & l_e \geqslant 0 & (e \in E), \end{array} \tag{11}$$

is equal to the cone generated by the following vectors:

$$\begin{array}{lll} \text{(i)} & (-\chi^{\rho(X)}; \chi^{\delta(X)}) & (X \subseteq V), \\ \text{(ii)} & (\varepsilon_i; 0) & (i = 1, ..., k), \\ \text{(iii)} & (0; \varepsilon_e) & (e \in E). \end{array} \tag{12}$$

Note that (11)(i) just means that $-b_i$ is a lower bound for the distance from r_i to s_i, taking l as a length function. So the statement is equivalent to: if (4), (5), or (6) holds, then for any "length function" $l: E \to \mathbb{Q}_+$, there exist subsets $X_1, ..., X_t$ of V and rationals $\mu_1, ..., \mu_t \geqslant 0$, so that

$$\begin{array}{lll} \text{(i)} & \sum (\mu_j \mid j = 1, ..., t;\ i \in \rho(X_j)) \geqslant \text{dist}_l(r_i, s_i) & (i = 1, ..., k), \\ \text{(ii)} & \sum (\mu_j \mid j = 1, ..., t;\ e \in \delta(X_j)) \leqslant l_e & (e \in E). \end{array} \tag{13}$$

[Here, dist_l denotes the distance, taking l as a length function. Note that equality in (i) can be derived from (ii).]

Now Karzanov [4] showed that if (4) holds, and if l is integral, we can take the μ_j half-integral. In fact, he showed that if l is integral so that each circuit of G has an even length, we can take the μ_j integral (thus extending work of Hu [2] and Seymour [13]). Equivalently, if G is bipartite and (4) holds, then there exist pairwise edge-disjoint cuts $\delta(X_1), ..., \delta(X_t)$ so that for each $i = 1, ..., k$, the distance from r_i to s_i is equal to the number of cuts $\delta(X_j)$ separating r_i and s_i. (The equivalence follows in one direction by taking $l_e = 1$ for each edge e, and in the other direction by replacing each edge e of length l_e by a path consisting of l_e edges.)

The theorem to be proved in this paper is similar, but now with respect to Okamura's condition (5) instead of Lomonosov's condition (4). Note that in a similar way as above, a fractional version of Okamura's theorem can be derived from our theorem. (For more on the duality of path and cut packing, see Karzanov [5].)

Professor A. V. Karzanov communicated to me that a similar theorem with respect to Seymour's condition (6) can be derived from Seymour [14].

3. Proof of the Theorem

Suppose that the theorem is not true, and let G be a counterexample with

$$\sum_{F \neq O,\, I} 2^{e(F)} \quad \text{as small as possible,} \tag{14}$$

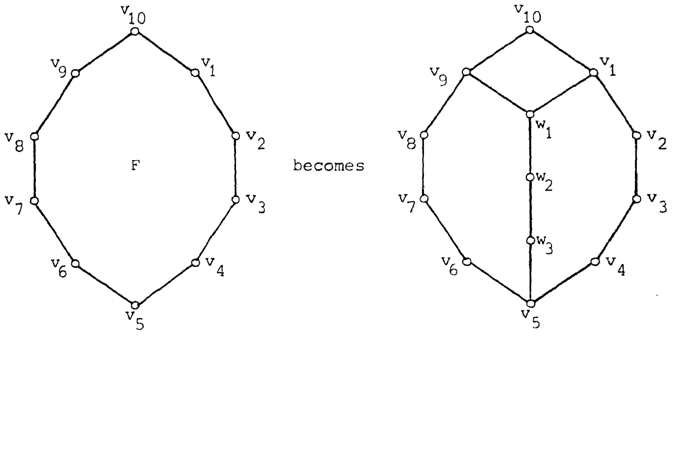
where the sum ranges over all faces $F \neq O, I$, and where $e(F)$ denotes the number of edges surrounding F. We may assume that O is the unbounded face.

G has no multiple edges: otherwise, either the circuit C formed by them is a face, in which case we can delete one of the edges, thereby decreasing sum (14), or C contains edges both in its interior and in its exterior, in which case the graph formed by C and its interior or the graph formed by C and its exterior yields a counterexample with smaller sum (14).

We first show:

Claim 1. *Each face $F \neq O, I$ forms a quadrangle* (*i.e.*, $e(F) = 4$).

Proof of Claim 1. Let F be some face forming a k-gon, with $k \neq 4$. Since G is bipartite and has no parallel edges, k is even and $k \geqslant 6$. We make a counterexample with a smaller sum than (14) as follows. Let $v_1, ..., v_k$ be the vertices surrounding F. Add, in the interior of F, new vertices $w_1, ..., w_{(1/2)k-2}$ and new edges $\{v_1, w_1\}$, $\{v_{k-1}, w_1\}$, $\{w_i, w_{i+1}\}$ ($i = 1, ..., \frac{1}{2}k - 3$), and $\{w_{(1/2)k-2}, v_{(1/2)k}\}$. E.g., for $k = 10$,



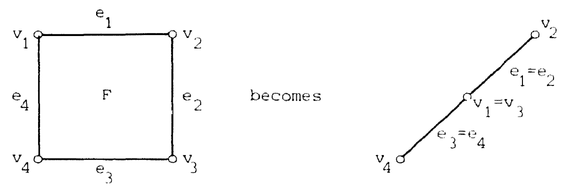
(15)

Note that this modification does not change the distance between any two vertices of the original graph. Therefore, after this modification we have again a counterexample to the theorem, with, however, a smaller sum than (14) (since $2^k > 2^{k-2} + 2^{k-2} + 2^4$), contradicting our assumption. ∎

Next we show:

CLAIM 2. *Let F be a face, with $F \neq O, I$, and let $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$, $e_3 = \{v_3, v_4\}$, $e_4 = \{v_4, v_1\}$ be the four edges surrounding F. Then there exist vertices v, w, with $v, w \in O$ or $v, w \in I$, and a shortest path from v to w which uses both e_1 and e_2.*

Proof of Claim 2. Suppose no such v, w exist. Identify v_1 and v_3, e_1 and e_2, and e_3 and e_4. So



(16)

After this modification, all distances between vertices v, w on O and between vertices v, w on I, are unchanged. Hence, the new graph is again a counterexample. However, the sum (14) has decreased, contradicting our assumption. ∎

Now we define *dual paths* $Q_1, ..., Q_t$, i.e., paths (including circuits) in the (planar) dual graph of G. These dual paths are determined by the following properties: each edge of the graph occurs exactly once in $Q_1, ..., Q_t$; if $F(\neq O, I)$ is surrounded by the edges e_1, e_2, e_3, e_4 (in this order), then e_1, F, e_3 (or e_3, F, e_1) will occur in exactly one of the Q_j; the faces O and I only occur as beginning or end faces in $Q_1, ..., Q_t$.

More precisely, $Q_1, ..., Q_t$ are all possible sequence of the form

$$(F_0, e_1, F_1, e_2, ..., F_{k-1}, e_k, F_k) \tag{17}$$

satisfying: (i) for $i = 1, ..., k$: e_i is an edge separating the faces F_{i-1} and F_i; (ii) for $i = 1, ..., k-1$: $F_i \notin \{O, I\}$ and e_i and e_{i+1} are opposite edges of F_i; (iii) either $F_0 = F_k \notin \{O, I\}$ and e_1 and e_k are opposite edges of F_0, or $F_0, F_k \in \{O, I\}$; (iv) $k \geqslant 1$. If $F_0 = F_k \notin \{O, I\}$, we identify all possible sequences obtained from (17) by cyclically shifting it or by reversing it. If $F_0, F_k \in \{O, I\}$, we identify (17) with its reverse. Here edge e is said to *separate* faces F and F' if F and F' are the faces incident to e (possibly $F = F'$). Clearly, in the way described the edges of G are partitioned into dual paths and circuits.

Consider now some fixed Q_g, represented by (17). Let for each $i=1, ..., k$, v_i and w_i be vertices so that $e_i=\{v_i, w_i\}$ and so that if we would orient the edges surrounding F_i clockwise, then e_i is oriented from v_i to w_i. Then $f_i := \{v_i, v_{i+1}\}$ and $g_i := \{w_i, w_{i+1}\}$ are also edges of G $(i=1, ..., k-1)$. So

$$(v_1, f_1, v_2, f_2, ..., v_{k-1}, f_{k-1}, v_k) \tag{18}$$

is the path along Q_g "on the right side," and

$$(w_1, g_1, w_2, g_2, ..., w_{k-1}, g_{k-1}, w_k) \tag{19}$$

is the path along Q_g "on the left side."

CLAIM 3. *For all* $i, j \in \{1, ..., k\}$: $\text{dist}(v_i, v_j) = \text{dist}(w_i, w_j)$, *where* dist *denotes distance.*

Proof of Claim 3. Suppose to the contrary that $\text{dist}(v_i, v_j) \neq \text{dist}(w_i, w_j)$ for some i, j. Choose such i, j so that $i<j$ and $j-i$ is as small as possible. By symmetry, we may assume that $\text{dist}(v_i, v_j) < \text{dist}(w_i, w_j)$. As G is bipartite, $j-i \geqslant \text{dist}(w_i, w_j) \geqslant \text{dist}(v_i, v_j)+2 \geqslant 2$.

Let

$$(v_i, \sigma, v_j) \tag{20}$$

be a shortest v_i-v_j-path, for some string σ. Since $\text{dist}(w_i, w_j) \geqslant \text{dist}(v_i, v_j)+2$, it follows that

$$(w_i, e_i, v_i, \sigma, v_j, e_j, w_j) \tag{21}$$

is a shortest w_i-w_j-path. Consider the circuit (see Fig. 1)

$$C := (v_i, \sigma, v_j, f_{j-1}, v_{j-1}, ..., f_{i+1}, v_{i+1}, f_i, v_i). \tag{22}$$

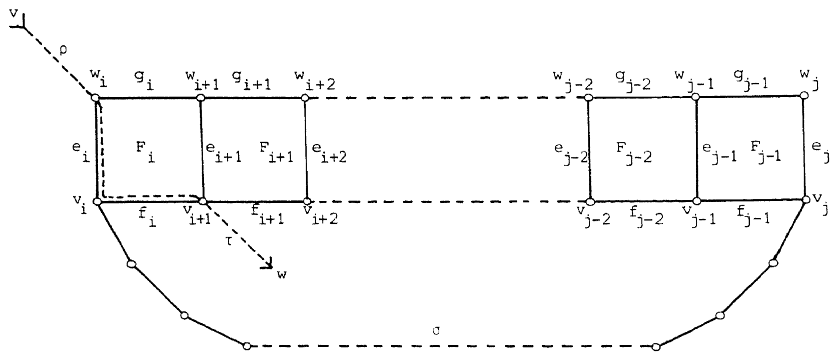


FIGURE 1

C is a simple circuit, i.e., no vertex occurs twice in (22), except for the beginning and end vertex. Indeed, all vertices in (20) are distinct, as it is a shortest path. Moreover, all vertices $v_i, v_{i+1}, \ldots, v_j$ are distinct, except possibly $v_i = v_j$: if $v_p = v_q$ with $i \leqslant p < q \leqslant j$, then $\text{dist}(v_p, v_q) = 0 < \text{dist}(w_p, w_q)$ (since G has no parallel edges), and hence, by the minimality of $j-i$, $q-p \geqslant j-i$; that is, $p = i$ and $q = j$. Suppose finally, $\sigma = (\sigma', v_q, \sigma'')$ for some strings σ', σ'' and $i+1 \leqslant q \leqslant j-1$. Then $\text{dist}(v_i, v_q) + \text{dist}(v_q, v_j) = \text{dist}(v_i, v_j)$ (as v_q is on the shortest $v_i - v_j$-path (20), and hence, $\text{dist}(v_i, v_q) + \text{dist}(v_q, v_j) = \text{dist}(v_i, v_j) < \text{dist}(w_i, w_j) \leqslant \text{dist}(w_i, w_q) + \text{dist}(w_q, w_j)$. Therefore, $\text{dist}(v_i, v_q) < \text{dist}(w_i, w_q)$ or $\text{dist}(v_q, v_j) < \text{dist}(w_q, w_j)$, contradicting the minimality of $j-i$.

By Claim 2, there exist vertices v and w, either both on O or both on I, and a shortest $v-w$-path P with

$$P = (v, \rho, w_i, e_i, v_i, f_i, v_{i+1}, \tau, w), \tag{23}$$

where ρ and τ are strings. Hence, the path

$$P' := (v, \rho, w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau, w) \tag{24}$$

is also a shortest $v-w$-path. Since $O, I \notin \{F_i, \ldots, F_{j-1}\}$ (as $1 \leqslant i < j \leqslant k$), we are in one of the following four cases (as either both v and w are enclosed by C (Cases 1 and 2) or not (Cases 3 and 4)).

Case 1. (v, ρ) and (v_i, σ, v_j) have a vertex in common, say u,

$$\begin{aligned} (v, \rho) &= (\rho', u, \rho''), \\ (v_i, \sigma, v_j) &= (\sigma', u, \sigma''), \end{aligned} \tag{25}$$

for (possibly empty) strings ρ', ρ'', σ', σ''. Then

$$(\rho', u, (\sigma')^{-1}, e_i, w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau, w) \tag{26}$$

also would be a shortest $v-w$-path, since (w_i, e_i, σ', u) is a shortest $w_i - u$-path (as it is part of (21)). But then

$$(\rho', u, (\sigma')^{-1}, f_i, v_{i+1}, \tau, w) \tag{27}$$

would be an even shorter $v-w$-path, which is a contradiction.

Case 2. (v, ρ) contains one of the edges $e_{i+1}, \ldots, e_{j-1}$, say $(v, \rho) = (\rho', v_p, e_p, w_p, \rho'')$ for some p with $i+1 \leqslant p \leqslant j-1$ and certain (possibly empty) strings ρ', ρ''. Substitution in P gives:

$$P = (\rho', v_p, e_p, w_p, \rho'', w_i, e_i, v_i, f_i, v_{i+1}, \tau, w). \tag{28}$$

Since P is a shortest $v-w$-path, it follows that $\text{dist}(w_p, w_i) < \text{dist}(v_p, v_i)$, contradicting the minimality of $j-i$.

Case 3. (τ, w) and (v_i, σ, v_j) have a vertex in common, say u,

$$(\tau, w) = (\tau', u, \tau''),$$
$$(v_i, \sigma, v_j) = (\sigma', u, \sigma''), \tag{29}$$

for (possibly empty) strings τ', τ'', σ', σ''. So

$$(w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau', u) \tag{30}$$

is not longer than

$$(w_i, e_i, \sigma', u) \tag{31}$$

(since (30) is part of the shortest path P'). Hence, substituting (31) by (30) in (21),

$$(w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau', u, \sigma'', v_j, e_j, w_j) \tag{32}$$

is a shortest $w_i - w_j$-path. In particular, $\text{dist}(v_{i+1}, v_j) < \text{dist}(w_{i+1}, w_j)$, contradicting the minimality of $j-i$.

Case 4. (τ, w) contains one of the edges $e_{i+1}, ..., e_{j-1}$, say $(\tau, w) = (\tau', v_p, e_p, w_p, \tau'')$ for some p with $i+1 \leqslant p \leqslant j-1$ and certain (possibly empty) strings τ', τ''. Substitution in P gives:

$$P = (v, \rho, w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau', v_p, e_p, w_p, \tau''). \tag{33}$$

Since P is a shortest $v-w$-path, it follows that $\text{dist}(v_{i+1}, v_p) < \text{dist}(w_{i+1}, w_p)$, contradicting the minimality of $j-i$. ∎

A consequence of Claim 3 is that Q_g will have no self-intersections: if $F_i = F_j$ with $i \neq j$ and $i-j \neq k$, then $v_i = v_{j+1}$, $w_i \neq w_{j+1}$, or $v_{i+1} = v_j$, $w_{i+1} \neq w_j$, as one easily checks. This contradicts Claim 3.

Another consequence of Claim 3 is:

$$\text{no shortest path has more than one edge in common with } Q_g. \tag{34}$$

Next we show:

CLAIM 4. *Each Q_g connects O and I.*

Proof of Claim 4. Suppose Q_g does not connect O and I, for some $g = 1, ..., t$. Then Q_g connects O with O, or connects I with I, or is a circuit. That is, the edges in Q_g form a cut $\delta(X)$, for some $X \subseteq V$.

I. We first show for each $v, w \in V$ that for each $v-w$-path P there exists a $v-w$-path P' so that

$$\text{length}(P') - \text{int}(P', Q_g) \leqslant \text{length}(P) - \text{int}(P, Q_g), \quad \text{and}$$
$$\text{int}(P', Q_g) \leqslant 1, \tag{35}$$

where int(..., Q_g) denotes the number of edges in ... in common with Q_g. This is shown by induction on length (P). If $\text{int}(P, Q_g) \geqslant 2$, there exist i, j so that $P = (\rho, v_i, e_i, w_i, \sigma, w_j, e_j, v_j, \tau)$ for strings ρ, σ, τ, where σ does not have any edge in common with Q_g (we use the notation introduced before Claim 3; maybe v_i, v_j and w_i, w_j are interchanged). Since by Claim 3, $\text{dist}(w_i, w_j) = \text{dist}(v_i, v_j)$, there exists a $v-w$-path $\tilde{P}$ with $\text{length}(\tilde{P}) \leqslant \text{length}(P) - 2$ and $\text{int}(\tilde{P}, Q_g) \geqslant \text{int}(P, Q_g) - 2$. Applying the induction hypothesis to $\tilde{P}$ implies the statement above.

II. Now contract all edges ocurring in Q_g. This gives a smaller bipartite graph G'. For the new distance function dist′ in G' we have

$$\begin{aligned} \text{dist}'(v, w) &= \text{dist}(v, w) - 1, && \text{if } X \text{ separates } v \text{ and } w, \\ \text{dist}'(v, w) &= \text{dist}(v, w), && \text{otherwise.} \end{aligned} \tag{36}$$

To see this, it suffices to show that $\text{dist}'(v, w) \geqslant \text{dist}(v, w) - 1$ for all v, w (by the bipartiteness of G and G'). Let Π be a shortest $v-w$-path in G'. It corresponds to a $v-w$-path P in G with $\text{length}(P) - \text{int}(P, Q_g) = \text{length}(\Pi)$. Hence, by I above, there exists a $v-w$-path P' in G so that $\text{length}(P') - \text{int}(P', Q_g) \leqslant \text{length}(\Pi)$ and $\text{int}(P', Q_g) \leqslant 1$. Hence, $\text{dist}(v, w) \leqslant \text{length}(P') \leqslant \text{length}(\Pi) + 1 = \text{dist}'(v, w) + 1$.

By the minimal property of G, in G' there exist pairwise disjoint cuts $\delta(X_1), ..., \delta(X_{t'})$ so that for all pairs of vertices v, w both on O or both on I:

$$\text{dist}'(v, w) = |\{i = 1, ..., t' \mid X_i \text{ separates } v \text{ and } w\}|. \tag{37}$$

So by (36), taking $X_{t'+1} := X$, in G we have for all such v, w:

$$\text{dist}(v, w) = |\{i = 1, ..., t' + 1 \mid X_i \text{ separates } v \text{ and } w\}|. \tag{38}$$

As $\delta(X_1), ..., \delta(X_{t'+1})$ are pairwise disjoint, G is not a counterexample to the theorem, contradicting our assumption. ∎

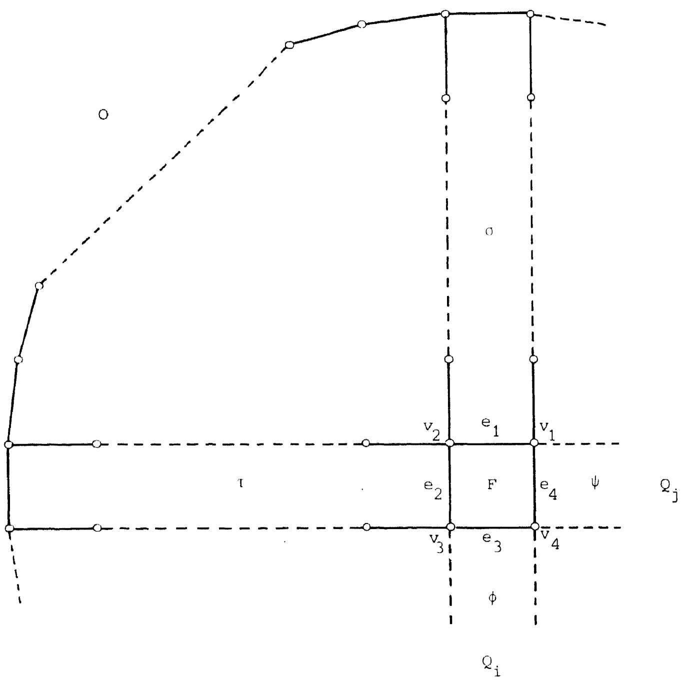
Our final claim will complete the counterexample:

CLAIM 5. *No two distinct Q_i and Q_j have a face $F \neq O, I$ in common.*

Proof of Claim 5. Suppose to the contrary

$$\begin{aligned} Q_i &= (O, \sigma, F, \varphi), \\ Q_j &= (O, \tau, F, \psi), \end{aligned} \tag{39}$$

for strings σ, φ, τ, ψ, and face $F \neq O, I$ $(i \neq j)$. We may assume that σ and τ do not have a face in common (by taking (39) so that σ and τ have minimal length). This gives the following situation



(40)

We may assume that e_1 is the last symbol of σ and that e_2 is the last symbol of τ. By Claim 2, there exist vertices v, w, both on O or both on I, and a shortest $v-w$-path P using e_2 and e_3:

$$P = (v, \pi, v_2, e_2, v_3, e_3, v_4, \rho, w). \tag{41}$$

As P is a shortest $v-w$-path, with $v, w \in O$ or $v, w \in I$, P has at most one edge in common with each of the Q_g $(g = 1, ..., k)$ (by (34)). Since P crosses both Q_i and Q_j at F, while the vertex v_2 is contained in the set of vertices enclosed by the dual circuit $(O, \sigma, F, \tau^{-1}, O)$, P should also have its beginning vertex v inside of this circuit. So v is on O, and hence also w is on O.

Since P has exactly one edge in common with Q_i, it follows that P is homotopic (in the space obtained from the euclidean plane by deleting the interiors of O (=unbounded face) and I) to the $v-w$-path P' which follows the boundary of O and which contains the first edge of Q_i. Similarly, P is homotopic to the $v-w$-path P'' which follows the boundary of O and which contains the first edge of Q_j. Since v is inside of the circuit $(O, \sigma, F, \tau^{-1}, O)$, while w is outside of it, P' is not homotopic to P'', a contradiction. ∎

Claim 5 implies that there are no faces other than O and I (any other face would belong to two different Q_i and Q_j). So G is a simple circuit, for which the theorem trivially holds.

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